HOMEWORK 1 EXTRA EXERCISES

The point of these exercises is to review some facts that we came across in class. It seems that most of you believe these facts and are willing to use them, but are not sure why they are true.

The goal is

(1) to convince yourselves that they are in fact true and learn why,
(2) to go back and review some basic facts and techniques that go into showing them.

The following exercise is useful because you will review the definition of $\ln x$, the definition of inverse functions, how exponents work, and the definition of one-to-one functions. (If you have trouble remembering any of these, go back and read the relevant sections in your book.)

Exercise 1. Show that for $a, b \in (0, \infty)$,

$$\ln a - \ln b = \ln \frac{a}{b}.$$

Hint: Recall the definition of $\ln x$ as the inverse of e^x, show that $e^{\ln a - \ln b} = e^{\ln \frac{a}{b}}$, and use the fact that e^x is one-to-one to draw the desired conclusion.

The following exercise is useful because in the process of doing it you will review differentiation (chain rule and quotient rule), the basic trigonometry relation relating $\sin x$ and $\cos x$ and other trigonometric relations you can deduce from it, and the definition of inverse functions. (If you have trouble remembering any of these, go back and read the relevant sections in your book.) Now you will beautifully combine all of these to compute the derivative of $\tan^{-1} x$. I know it's a challenging exercise, but with the step by step guidance and hints along the way, it should be very manageable, and if you get through it you will make me so happy, because we will have reviewed so many concepts in the process.

Exercise 2. Show that

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

Here are the steps. Do each one of them: (a), (b), (c) are unrelated to each other, but they all get combined in part (d) to prove the result.

(a) Suppose that f and g are two differentiable functions that are inverses of each other. Show that

$$g'(x) = \frac{1}{f'(g(x))}$$

(if $f'(g(x)) \neq 0$, of course. Question: could $f'(g(x))$ be 0? Think of an example or why not)

Hint: If f and g are inverses of each other, then by definition $f(g(x)) = x$ and $g(f(x)) = x$. Apply the chain rule to the first one of these expressions.

(b) Show that

$$\frac{d}{dx} \tan x = \sec^2 x,$$

where recall that, by definition $\sec x = \frac{1}{\cos x}$.

Hint: Use the quotient rule on $\tan x = \frac{\sin x}{\cos x}$ and the identity $\cos^2 x + \sin^2 x = 1$.

Note: This last identity is true because, as I reminded you in class, $(\cos x, \sin x)$ are the coordinates of the point on the unit circle corresponding to the angle of measure x, and all points (x, y) on the unit circle, by definition satisfy the equation $x^2 + y^2 = 1$. Alternatively, you can deduce it using the Pythagorean theorem. If you don't believe it, think about it. You can assume it for this problem, though.

(c) Show that

$$\sec^2(\tan^{-1} x) = 1 + x^2.$$

Hint: Set $\tan^{-1} x = y$, so that $\tan y = x$ and show that $1 + \tan^2 y = \sec^2 y$.

(d) Combine (a), (b) and (c) to draw the conclusion.
Hint: Note that by definition $\tan x$ and $\tan^{-1} x$ are inverse functions so that's how you can use (a).